Distance-regular graphs, pseudo primitive idempotents, and the Terwilliger algebra

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Abstract

Let Γ denote a distance-regular graph with diameter $D \geq 3$, intersection numbers a_i, b_i, c_i and Bose-Mesner algebra $\mathbf{M}$. For $\theta \in \mathbb{C} \cup \infty$ we define a 1 dimensional subspace of $\mathbf{M}$ which we call $\mathbf{M}(\theta)$. If $\theta \in \mathbb{C}$ then $\mathbf{M}(\theta)$ consists of those Y in $\mathbf{M}$ such that $(A - \theta I)Y \in \mathbb{C}A_D$, where A (resp. A_D) is the adjacency matrix (resp. Dth distance matrix) of Γ. If $\theta = \infty$ then $\mathbf{M}(\theta) = \mathbb{C}A_D$. By a *pseudo primitive idempotent* for θ we mean a nonzero element of $\mathbf{M}(\theta)$. We use these as follows. Let X denote the vertex set of Γ and fix $x \in X$. Let $\mathbf{T}$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, $E_0^*, E_1^*, \cdots, E_D^*$, where E_i^* denotes the projection onto the ith subconstituent of Γ with respect to x. $\mathbf{T}$ is called the Terwilliger algebra. Let W denote an irreducible $\mathbf{T}$-module. By the *endpoint* of W we mean $\min\{i | E_i^* W \neq 0\}$. W is called *thin* whenever $\dim(E_i^* W) \leq 1$ for $0 \leq i \leq D$. Let $V = \mathbb{C}^X$ denote the standard $\mathbf{T}$-module. Fix $0 \neq v \in E_1^* V$ with v orthogonal to the all 1's vector. We define $(\mathbf{M}; v) := \{P \in \mathbf{M} | Pv \in E_D^* V\}$. We show the following are equivalent: (i) $\dim(\mathbf{M}; v) \geq 2$; (ii) v is contained in a thin irreducible $\mathbf{T}$-module with endpoint 1. Suppose (i), (ii) hold. We show $(\mathbf{M}; v)$ has a basis J, E where J has all entries 1 and E is defined as follows. Let W denote the $\mathbf{T}$-module which satisfies (ii). Observe $E_1^* W$ is an eigenspace for $E_1^* A E_1^*$; let η denote the corresponding eigenvalue. Define $\tilde{\eta} = -1 - b_1(1+\eta)^{-1}$ if $\eta \neq -1$ and $\tilde{\eta} = \infty$ if $\eta = -1$. Then E is a pseudo primitive idempotent for $\tilde{\eta}$.

Keywords: distance-regular graph, pseudo primitive idempotents, subconstituent algebra, Terwilliger algebra.
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1 Introduction

Let Γ denote a distance-regular graph with diameter $D \geq 3$, intersection numbers a_i, b_i, c_i, Bose-Mesner algebra $\mathbf{M}$ and path-length distance function ∂ (see Section 2 for formal definitions). In order to state our main theorems we make a few comments. Let X denote the vertex set of Γ. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We endow V with the Hermitean inner product $\langle\, , \rangle$ satisfying $\langle u, v\rangle = u^t\overline{v}$ for all $u, v \in V$. For each $y \in X$ let $\hat{y}$ denote the vector in V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} | y \in X\}$ is an orthonormal basis for V. Fix $x \in X$. For $0 \leq i \leq D$ let E_i^* denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ which has yy entry 1 (resp. 0) whenever $\partial(x, y) = i$ (resp. $\partial(x, y) \neq i$). We observe E_i^* acts on V as the projection onto the ith subconstituent of Γ with respect to x. For $0 \leq i \leq D$ define $s_i = \sum \hat{y}$, where the sum is over all vertices $y \in X$ such that $\partial(x, y) = i$. We observe $s_i \in E_i^*V$. Let v denote a nonzero vector in E_1^*V which is orthogonal to s_1. We define

$$(\mathbf{M}; v) := \{P \in \mathbf{M} \mid Pv \in E_D^*V\}.$$

We observe $(\mathbf{M}; v)$ is a subspace of $\mathbf{M}$. We consider the dimension of $(\mathbf{M}; v)$.

We first observe $(\mathbf{M}; v) \neq 0$. To see this, let J denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ which has all entries 1. It is known J is contained in $\mathbf{M}$ [2, p. 64]. In fact $J \in (\mathbf{M}; v)$; the reason is $Jv = 0$ since v is orthogonal to s_1. Apparently $(\mathbf{M}; v)$ is nonzero so it has dimension at least 1. We now consider when does $(\mathbf{M}; v)$ have dimension at least 2? To answer this question we recall the Terwilliger algebra. Let $\mathbf{T}$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where A denotes the adjacency matrix of Γ. The algebra $\mathbf{T}$ is known as the *Terwilliger* algebra (or *subconstituent* algebra) of Γ with respect to x [19, 20, 21]. By a $\mathbf{T}$-*module* we mean a subspace $W \subseteq V$ such that $\mathbf{T}W \subseteq W$. Let W denote a $\mathbf{T}$-module. We say W is *irreducible* whenever $W \neq 0$ and W does not contain a $\mathbf{T}$-module other than 0 and W. Let W denote an irreducible $\mathbf{T}$-module. By the *endpoint* of W we mean the minimal integer i $(0 \leq i \leq D)$ such that $E_i^*W \neq 0$. We say W is *thin* whenever E_i^*W has dimension at most 1 for $0 \leq i \leq D$. We now state our main theorem.

Theorem 1.1. *Let v denote a nonzero vector in E_1^*V which is orthogonal to s_1. Then the following (i), (ii) are equivalent.*

(i) $(\mathbf{M}; v)$ *has dimension at least* 2.

(ii) v *is contained in a thin irreducible* $\mathbf{T}$*-module with endpoint 1.*

Suppose (i),(ii) hold above. Then $(\mathbf{M}; v)$ *has dimension exactly* 2.

With reference to Theorem 1.1, suppose for the moment that (i), (ii) hold. We find a basis for $(\mathbf{M}; v)$. To describe our basis we need some notation. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of A, and for $0 \leq i \leq D$ let E_i denote the primitive idempotent of $\mathbf{M}$ associated with θ_i. We recall E_i satisfies $(A - \theta_i I)E_i = 0$. We introduce a type of element in $\mathbf{M}$ which generalizes the $E_0, E_1, \ldots, E_D$. We call this type of element a *pseudo primitive idempotent* for Γ. In order to define the pseudo primitive idempotents, we first define for each $\theta \in \mathbb{C} \cup \infty$ a subspace of $\mathbf{M}$ which we call $\mathbf{M}(\theta)$. For $\theta \in \mathbb{C}$, $\mathbf{M}(\theta)$ consists of those elements Y of $\mathbf{M}$ such that $(A - \theta I)Y \in \mathbb{C}A_D$, where A_D is the Dth distance matrix of Γ. We define $\mathbf{M}(\infty) = \mathbb{C}A_D$. We show $\mathbf{M}(\theta)$ has dimension 1 for all $\theta \in \mathbb{C} \cup \infty$. Given distinct θ, θ' in $\mathbb{C} \cup \infty$, we show $\mathbf{M}(\theta) \cap \mathbf{M}(\theta') = 0$. For $0 \leq i \leq D$ we show $\mathbf{M}(\theta_i) = \mathbb{C}E_i$. Let $\theta \in \mathbb{C} \cup \infty$. By a *pseudo primitive idempotent* for θ, we mean a nonzero element of $\mathbf{M}(\theta)$. Before proceeding we define an involution on $\mathbb{C} \cup \infty$. For $\eta \in \mathbb{C} \cup \infty$ we define

$$\widetilde{\eta} = \begin{cases} \infty & \text{if } \eta = -1, \\ -1 & \text{if } \eta = \infty, \\ -1 - \frac{b_1}{1+\eta} & \text{if } \eta \neq -1, \eta \neq \infty. \end{cases}$$

We observe $\widetilde{\widetilde{\eta}} = \eta$ for $\eta \in \mathbb{C} \cup \infty$. Let W denote a thin irreducible $\mathbf{T}$-module with endpoint 1. Observe $E_1^* W$ is a one dimensional eigenspace for $E_1^* A E_1^*$; let η denote the corresponding eigenvalue. We call η the *local eigenvalue* of W.

Theorem 1.2. *Let* v *denote a nonzero vector in* $E_1^* V$ *which is orthogonal to* s_1*. Suppose* v *satisfies the equivalent conditions (i), (ii) in Theorem 1.1. Let* W *denote the* $\mathbf{T}$*-module from part (ii) of that theorem and let* η *denote the local eigenvalue for* W*. Let* E *denote a pseudo primitive idempotent for* $\widetilde{\eta}$*. Then* J, E *form a basis for* $(\mathbf{M}; v)$*.*

We comment on when the scalar $\widetilde{\eta}$ from Theorem 1.2 is an eigenvalue of Γ. Let W denote a thin irreducible $\mathbf{T}$-module with endpoint 1 and local

eigenvalue η. It is known $\widetilde{\theta}_1 \leq \eta \leq \widetilde{\theta}_D$ [18, Theorem 1]. If $\eta = \widetilde{\theta}_1$ then $\widetilde{\eta} = \theta_1$. If $\eta = \widetilde{\theta}_D$ then $\widetilde{\eta} = \theta_D$. We show that if $\widetilde{\theta}_1 < \eta < \widetilde{\theta}_D$ then $\widetilde{\eta}$ is not an eigenvalue of Γ.

The paper is organized as follows. In section 2 we give some preliminaries on distance-regular graphs. In section 3 and section 4 we review some basic results on the Terwilliger algebra and its modules. We prove Theorem 1.1 in section 5. In section 6 we discuss pseudo primitive idempotents. In section 7 we discuss local eigenvalues. We prove Theorem 1.2 in section 8.

2 Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [2] or Brouwer, Cohen, and Neumaier [4] for more background information.

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitean inner product $\langle \, , \, \rangle$ which satisfies $\langle u, v \rangle = u^t \overline{v}$ for all $u, v \in V$, where t denotes transpose and $-$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂ and diameter $D := \max\{\partial(x,y) | x, y \in X\}$. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X | \partial(x,z) = i, \partial(z,y) = j\}| \tag{2.1}$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* for Γ. Observe $p^h_{ij} = p^h_{ji}$ $(0 \leq h, i, j \leq D)$. We abbreviate $c_i := p^i_{1i-1}$ $(1 \leq i \leq D)$, $a_i := p^i_{1i}$ $(0 \leq i \leq D)$, $b_i := p^i_{1i+1}$ $(0 \leq i \leq D-1)$, $k_i := p^0_{ii}$

$(0 \leq i \leq D)$, and for convenience we set $c_0 := 0$ and $b_D := 0$. Note that $b_{i-1}c_i \neq 0$ $(1 \leq i \leq D)$.

For the rest of this paper we assume $\Gamma = (X, R)$ is distance-regular with diameter $D \geq 3$. By (2.1) and the triangle inequality,

$$p^h_{i1} = 0 \quad \text{if } |h-i| > 1 \quad (0 \leq h, i \leq D), \tag{2.2}$$

$$p^1_{ij} = 0 \quad \text{if } |i-j| > 1 \quad (0 \leq i, j \leq D). \tag{2.3}$$

Observe Γ is regular with valency $k = k_1 = b_0$, and that $k = c_i + a_i + b_i$ for $0 \leq i \leq D$. By [4, p. 127] we have

$$k_{i-1}b_{i-1} = k_i c_i \qquad (1 \leq i \leq D). \tag{2.4}$$

We recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ which has yz entry

$$(A_i)_{yz} = \begin{cases} 1 & \text{if } \partial(y,z) = i \\ 0 & \text{if } \partial(y,z) \neq i \end{cases} \qquad (y, z \in X).$$

We call A_i the *ith distance matrix* of Γ. For notational convenience we define $A_i = 0$ for $i < 0$ and $i > D$. Observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^D A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \leq i \leq D)$; (aiv) $A_i^t = A_i$ $(0 \leq i \leq D)$, (av) $A_iA_j = \sum_{h=0}^D p^h_{ij} A_h$ $(0 \leq i, j \leq D)$, where I denotes the identity matrix and J denotes the all ones matrix. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. Let $\mathbf{M}$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. Using (ai)–(av) we find $A_0, A_1, \cdots, A_D$ form a basis of $\mathbf{M}$. We call $\mathbf{M}$ the *Bose-Mesner algebra* of Γ. By [2, p. 59, p. 64], $\mathbf{M}$ has a second basis E_0, E_1, $\cdots$, E_D such that (ei) $E_0 = |X|^{-1}J$; (eii) $\sum_{i=0}^D E_i = I$; (eiii) $\overline{E_i} = E_i$ $(0 \leq i \leq D)$; (eiv) $E_i^t = E_i$ $(0 \leq i \leq D)$; (ev) $E_iE_j = \delta_{ij}E_i$ $(0 \leq i, j \leq D)$. We call E_0, E_1, $\cdots$, E_D the *primitive idempotents* for Γ. Since E_0, E_1, $\cdots$, E_D form a basis for $\mathbf{M}$ there exists complex scalars $\theta_0, \theta_1, \cdots, \theta_D$ such that $A = \sum_{i=0}^D \theta_i E_i$. By this and (ev) we find $AE_i = \theta_i E_i$ for $0 \leq i \leq D$. Using (aiii) and (eiii) we find each of θ_0, θ_1, $\cdots$, θ_D is a real number. Observe θ_0, θ_1, $\cdots$, θ_D are mutually distinct since A generates $\mathbf{M}$. By [2, p.197] we have $\theta_0 = k$ and $-k \leq \theta_i \leq k$ for $0 \leq i \leq D$. Throughout this paper, we assume E_0, E_1, $\cdots$, E_D are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_D$. We call θ_i the *ith eigenvalue* of Γ.

We recall some polynomials. To motivate these we make a comment. Setting $i = 1$ in (av) and using (2.2),

$$AA_j = b_{j-1}A_{j-1} + a_jA_j + c_{j+1}A_{j+1} \qquad (0 \le j \le D-1), \tag{2.5}$$

where $b_{-1} = 0$. Let λ denote an indeterminate and let $\mathbb{C}[\lambda]$ denote the $\mathbb{C}$-algebra consisting of all polynomials in λ which have coefficients in $\mathbb{C}$. Let $f_0, f_1, \cdots, f_D$ denote the polynomials in $\mathbb{C}[\lambda]$ which satisfy $f_0 = 1$ and

$$\lambda f_j = b_{j-1}f_{j-1} + a_jf_j + c_{j+1}f_{j+1} \qquad (0 \le j \le D-1), \tag{2.6}$$

where $f_{-1} = 0$. For $0 \le j \le D$ the degree of f_j is exactly j. Comparing (2.5) and (2.6) we find $A_j = f_j(A)$.

3 The Terwilliger algebra

For the remainder of this paper we fix $x \in X$. For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ which has yy entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad (y \in X). \tag{3.1}$$

We call E_i^* the *ith dual idempotent of Γ with respect to x*. For convenience we define $E_i^* = 0$ for $i < 0$ and $i > D$. We observe (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le D)$, (iii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$, (iv) $E_i^*E_j^* = \delta_{ij}E_i^*$ $(0 \le i, j \le D)$. The E_i^* have the following interpretation. Using (3.1) we find

$$E_i^*V = \mathrm{span}\{\hat{y} | y \in X, \quad \partial(x,y) = i\} \qquad (0 \le i \le D).$$

By this and since $\{\hat{y} | y \in X\}$ is an orthonormal basis for V,

$$V = E_0^*V + E_1^*V + \cdots + E_D^*V \qquad \text{(orthogonal direct sum)}.$$

For $0 \le i \le D$, E_i^* acts on V as the projection onto E_i^*V. We call E_i^*V the *ith subconstituent of Γ with respect to x*. For $0 \le i \le D$ we define $s_i = \sum \hat{y}$, where the sum is over all vertices $y \in X$ such that $\partial(x,y) = i$. We observe $s_i \in E_i^*V$.

Let $\mathbf{T} = \mathbf{T}(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, E_0^*, E_1^*, $\cdots$, E_D^*. The algebra $\mathbf{T}$ is semisimple but not commutative in general [19,

Lemma 3.4]. We call $\mathbf{T}$ the *Terwilliger algebra* (or *subconstituent algebra*) of Γ with respect to x. We refer the reader to [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24] for more information on the Terwilliger algebra. We will use the following facts. Pick any integers h, i, j $(0 \leq h, i, j \leq D)$. By [19, Lemma 3.2] we have $E_i^* A_h E_j^* = 0$ if and only if $p_{ij}^h = 0$. By this and (2.2), (2.3) we find

$$\begin{aligned} E_i^* A_h E_1^* &= 0 \quad \text{if} \quad |h-i| > 1 \qquad (0 \leq h, i \leq D), \qquad (3.2)\\ E_i^* A E_j^* &= 0 \quad \text{if} \quad |i-j| > 1 \qquad (0 \leq i, j \leq D). \qquad (3.3) \end{aligned}$$

Lemma 3.1. *The following (i), (ii) hold for* $0 \leq i \leq D$.

(i) $E_i^* J E_1^* = E_i^* A_{i-1} E_1^* + E_i^* A_i E_1^* + E_i^* A_{i+1} E_1^*$.

(ii) $A_i E_1^* = E_{i-1}^* A_i E_1^* + E_i^* A_i E_1^* + E_{i+1}^* A_i E_1^*$.

Proof. (i) Recall $J = \sum_{h=0}^{D} A_h$ so $E_i^* J E_1^* = \sum_{h=0}^{D} E_i^* A_h E_1^*$. Evaluating this using (3.2) we obtain the result.
(ii) Recall $I = \sum_{h=0}^{D} E_h^*$ so $A_i E_1^* = \sum_{h=0}^{D} E_h^* A_i E_1^*$. Evaluating this using (3.2) we obtain the result. □

Lemma 3.2. *For* $0 \leq i \leq D-1$ *we have*

$$E_{i+1}^* A_i E_1^* - E_i^* A_{i+1} E_1^* = \sum_{h=0}^{i} A_h E_1^* - \sum_{h=0}^{i} E_h^* J E_1^*. \qquad (3.4)$$

Proof. Evaluate each term in the right-hand side of (3.4) using Lemma 3.1 and simplify the result. □

Corollary 3.3. *Let v denote a vector in $E_1^* V$ which is orthogonal to s_1. Then for* $0 \leq i \leq D-1$ *we have*

$$E_{i+1}^* A_i v - E_i^* A_{i+1} v = \sum_{h=0}^{i} A_h v. \qquad (3.5)$$

Proof. Apply all terms of (3.4) to v and evaluate the result using $E_1^* v = v$ and $Jv = 0$. □

Lemma 3.4. *The following (i), (ii) hold for* $1 \leq i \leq D-1$.

(i) $E_{i+1}^* A E_i^* A_{i-1} E_1^* = c_i E_{i+1}^* A_i E_1^*$

(ii) $E^*_{i-1}AE^*_iA_{i+1}E^*_1 = b_iE^*_{i-1}A_iE^*_1$.

Proof. (i) For all $y, z \in X$, on either side the yz entry is equal to c_i if $\partial(x, y) = i + 1$, $\partial(x, z) = 1$, $\partial(y, z) = i$, and zero otherwise.
(ii) For all $y, z \in X$, on either side the yz entry is equal to b_i if $\partial(x, y) = i-1$, $\partial(x, z) = 1$, $\partial(y, z) = i$, and zero otherwise. □

Corollary 3.5. *Let v denote a vector in E^*_1V. Then the following (i), (ii) hold for $1 \leq i \leq D - 1$.*

(i) Suppose $E^*_iA_{i-1}v = 0$. Then $E^*_{i+1}A_iv = 0$.

(ii) Suppose $E^*_iA_{i+1}v = 0$. Then $E^*_{i-1}A_iv = 0$.

Proof. In Lemma 3.4(i),(ii) apply both sides to v and use $E^*_1v = v$. □

4 The modules of the Terwilliger algebra

Let $\mathbf{T}$ denote the Terwilliger algebra of Γ with respect to x. By a $\mathbf{T}$-*module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in \mathbf{T}$. Let W denote a $\mathbf{T}$-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no $\mathbf{T}$-modules other than 0 and W. Let W denote an irreducible $\mathbf{T}$-module. Then W is the orthogonal direct sum of the nonzero spaces among $E^*_0W, E^*_1W, \ldots, E^*_DW$ [19, Lemma 3.4]. By the *endpoint* of W we mean $\min\{i|0 \leq i \leq D, E^*_iW \neq 0\}$. By the *diameter* of W we mean $|\{i|0 \leq i \leq D, E^*_iW \neq 0\}| - 1$. We say W is *thin* whenever E^*_iW has dimension at most 1 for $0 \leq i \leq D$. There exists a unique irreducible $\mathbf{T}$-module which has endpoint 0 [10, Prop. 8.4]. This module is called V_0. For $0 \leq i \leq D$ the vector s_i is a basis for $E^*_iV_0$ [19, Lemma 3.6]. Therefore V_0 is thin with diameter D. The module V_0 is orthogonal to each irreducible $\mathbf{T}$-module other than V_0 [6, Lem. 3.3]. For more information on V_0 see [6, 10]. We will use the following facts.

Lemma 4.1. *[19, Lemma 3.9] Let W denote an irreducible $\mathbf{T}$-module with endpoint r and diameter d. Then*

$$E^*_iW \neq 0 \qquad (r \leq i \leq r + d). \tag{4.1}$$

Moreover

$$E^*_iAE^*_jW \neq 0 \quad \text{if } |i - j| = 1, \quad (r \leq i, j \leq r + d). \tag{4.2}$$

Lemma 4.2. *[6, Lemma 3.4] Let W denote a $\mathbf{T}$-module. Suppose there exists an integer i $(0 \leq i \leq D)$ such that $\dim(E_i^* W) = 1$ and $W = \mathbf{T} E_i^* W$. Then W is irreducible.*

Theorem 4.3. *[12, Lemma 10.1], [22, Theorem 11.1] Let W denote a thin irreducible $\mathbf{T}$-module with endpoint one, and let v denote a nonzero vector in $E_1^* W$. Then $W = \mathbf{M}v$. Moreover the diameter of W is $D-2$ or $D-1$.*

Theorem 4.4. *[12, Corollary 8.6, Theorem 9.8] Let v denote a nonzero vector in $E_1^* V$ which is orthogonal to s_1. Then the dimension of $\mathbf{M}v$ is $D-1$ or D. Suppose the dimension of $\mathbf{M}v$ is $D-1$. Then $\mathbf{M}v$ is a thin irreducible $\mathbf{T}$-module with endpoint 1 and diameter $D-2$.*

5 The proof of Theorem 1.1

We now give a proof of Theorem 1.1.

Proof. ((i) $\Longrightarrow$ (ii)) We show $\mathbf{M}v$ is a thin irreducible $\mathbf{T}$-module with endpoint 1. By Theorem 4.4 the dimension of $\mathbf{M}v$ is either $D-1$ or D. First assume the dimension of $\mathbf{M}v$ is equal to $D-1$. Then by Theorem 4.4, $\mathbf{M}v$ is a thin irreducible $\mathbf{T}$-module with endpoint 1. Next assume the dimension of $\mathbf{M}v$ is equal to D. The space $(\mathbf{M}; v)$ contains J and has dimension at least 2, so there exists $P \in (\mathbf{M}; v)$ such that J, P are linearly independent. From the construction $Pv \in E_D^* V$. Observe $Pv \neq 0$; otherwise the dimension of $\mathbf{M}v$ is not D. The elements $A_0, A_1, \ldots, A_D$ form a basis for $\mathbf{M}$. Therefore the elements $A_0 + A_1 + \cdots + A_i$ $(0 \leq i \leq D)$ form a basis for $\mathbf{M}$. Apparently there exist complex scalars ρ_i $(0 \leq i \leq D)$ such that $P = \sum_{i=0}^{D} \rho_i (A_0 + A_1 + \cdots + A_i)$. Recall $J = \sum_{h=0}^{D} A_h$. Subtracting a scalar multiple of J from P if necessary, we may assume $\rho_D = 0$. We consider Pv from two points of view. On one hand we have $Pv \in E_D^* V$. Therefore $E_D^* Pv = Pv$ and $E_i^* Pv = 0$ for $0 \leq i \leq D-1$. On the other hand using (3.5),

$$Pv = \sum_{i=0}^{D-1} \rho_i (E_{i+1}^* A_i v - E_i^* A_{i+1} v).$$

Combining these two points of view we find $Pv = \rho_{D-1} E_D^* A_{D-1} v$, $E_0^* A v = 0$, and

$$\rho_{i-1} E_i^* A_{i-1} v = \rho_i E_i^* A_{i+1} v \qquad (1 \leq i \leq D-1). \tag{5.1}$$

We mentioned $Pv \neq 0$; therefore $\rho_{D-1} \neq 0$ and $E^*_D A_{D-1} v \neq 0$. Applying Corollary 3.5(i) we find $E^*_i A_{i-1} v \neq 0$ for $1 \leq i \leq D$. We claim $E^*_i A_{i+1} v$ and $E^*_i A_{i-1} v$ are linearly dependent for $1 \leq i \leq D-1$. Suppose there exists an integer i $(1 \leq i \leq D-1)$ such that $E^*_i A_{i+1} v$ and $E^*_i A_{i-1} v$ are linearly independent. Then $E^*_i A_{i+1} v \neq 0$. Applying Corollary 3.5(ii) we find $E^*_j A_{j+1} v \neq 0$ for $i \leq j \leq D-1$. Using these facts and (5.1) we routinely find $\rho_j = 0$ for $i \leq j \leq D-1$. In particular $\rho_{D-1} = 0$ for a contradiction. We have now shown $E^*_i A_{i+1} v$ and $E^*_i A_{i-1} v$ are linearly dependent for $1 \leq i \leq D-1$. Observe $\mathbf{M}v$ is spanned by the vectors

$$(A_0 + A_1 + \cdots + A_i)v \qquad\qquad (0 \leq i \leq D-1).$$

By (3.5) and our above comments we find $\mathbf{M}v$ is contained in the span of

$$E^*_{i+1} A_i v \qquad\qquad (0 \leq i \leq D-1). \tag{5.2}$$

Since $\mathbf{M}v$ has dimension D we find $\mathbf{M}v$ is equal to the span of (5.2). Apparently $\mathbf{M}v$ is a $\mathbf{T}$-module. Moreover $\mathbf{M}v$ is irreducible by Lemma 4.2. Apparently $\mathbf{M}v$ is thin with endpoint 1.

((ii) $\Longrightarrow$ (i)) We show $(\mathbf{M}; v)$ has dimension at least 2. Since $J \in (\mathbf{M}; v)$ it suffices to exhibit an element $P \in (\mathbf{M}; v)$ such that J, P are linearly independent. Let W denote a thin irreducible $\mathbf{T}$-module which has endpoint 1 and contains v. By Theorem 4.3 we have $W = \mathbf{M}v$; also by Theorem 4.3 the diameter of W is $D-2$ or $D-1$. First suppose W has diameter $D-2$. Then W has dimension $D-1$. Consider the map $\sigma : \mathbf{M} \to V$ which sends each element P to Pv. The image of $\mathbf{M}$ under σ is $\mathbf{M}v$ and the kernel of σ is contained in $(\mathbf{M}; v)$. The image has dimension $D-1$ and $\mathbf{M}$ has dimension $D+1$ so the kernel has dimension 2. It follows $(\mathbf{M}; v)$ has dimension at least 2. Next assume W has diameter $D-1$. In this case $E^*_D W \neq 0$ by (4.1). Since $W = \mathbf{M}v$ there exists $P \in \mathbf{M}$ such that Pv is a nonzero element in $E^*_D W$. Now $P \in (\mathbf{M}; v)$. Observe P, J are linearly independent since $Pv \neq 0$ and $Jv = 0$. Apparently the dimension of $(\mathbf{M}; v)$ is at least 2.

Now assume (i), (ii) hold. We show the dimension of $(\mathbf{M}; v)$ is 2. To do this, we show the dimension of $(\mathbf{M}; v)$ is at most 2. Let H denote the subspace of $\mathbf{M}$ spanned by $A_0, A_1, \ldots, A_{D-2}$. We show H has 0 intersection with $(\mathbf{M}; v)$. By Theorem 4.4 the dimension of $\mathbf{M}v$ is at least $D-1$. Recall $\mathbf{M}$ is generated by A so the vectors $A^i v$ $(0 \leq i \leq D-2)$ are linearly independent. Apparently the vectors $A_i v$ $(0 \leq i \leq D-2)$ are linearly independent. For $0 \leq i \leq D-2$

the vector $A_i v$ is contained in $\sum_{h=0}^{D-1} E^*_h V$ by Lemma 3.1(ii); therefore $A_i v$ is orthogonal to $E^*_D V$. We now see the vectors $A_i v$ $(0 \leq i \leq D-2)$ are linearly independent and orthogonal to $E^*_D V$. It follows H has 0 intersection with $(\mathbf{M}; v)$. Observe H is codimension 2 in $\mathbf{M}$ so the dimension of $(\mathbf{M}; v)$ is at most 2. We conclude the dimension of $(\mathbf{M}; v)$ is 2. $\square$

6 Pseudo primitive idempotents

In this section we introduce the notion of a pseudo primitive idempotent.

Definition 6.1. For each $\theta \in \mathbb{C} \cup \infty$ we define a subspace of $\mathbf{M}$ which we call $\mathbf{M}(\theta)$. For $\theta \in \mathbb{C}$, $\mathbf{M}(\theta)$ consists of those elements Y of $\mathbf{M}$ such that $(A - \theta I)Y \in \mathbb{C}A_D$. We define $\mathbf{M}(\infty) = \mathbb{C}A_D$.

With reference to Definition 6.1, we will show each $\mathbf{M}(\theta)$ has dimension 1. To establish this we display a basis for $\mathbf{M}(\theta)$. We will use the following result.

Lemma 6.2. *Let Y denote an element of $\mathbf{M}$ and write $Y = \sum_{i=0}^{D} \rho_i A_i$. Let θ denote a complex number. Then the following (i), (ii) are equivalent.*

(i) $(A - \theta I)Y \in \mathbb{C}A_D$.

(ii) $\rho_i = \rho_0 f_i(\theta) k_i^{-1}$ *for* $0 \leq i \leq D$.

Proof. Evaluating $(A - \theta I)Y$ using $Y = \sum_{i=0}^{D} \rho_i A_i$ and simplifying the result using (2.5) we obtain

$$(A - \theta I)Y = \sum_{i=0}^{D} A_i (c_i \rho_{i-1} + a_i \rho_i + b_i \rho_{i+1} - \theta \rho_i),$$

where $\rho_{-1} = 0$ and $\rho_{D+1} = 0$. Observe by (2.4), (2.6) that $\rho_i = \rho_0 f_i(\theta) k_i^{-1}$ for $0 \leq i \leq D$ if and only if $c_i \rho_{i-1} + a_i \rho_i + b_i \rho_{i+1} = \theta \rho_i$ for $0 \leq i \leq D-1$. The result follows. $\square$

Corollary 6.3. *For $\theta \in \mathbb{C}$ the following is a basis for $\mathbf{M}(\theta)$.*

$$\sum_{i=0}^{D} f_i(\theta) k_i^{-1} A_i. \tag{6.1}$$

Proof. Immediate from Lemma 6.2. $\square$

Corollary 6.4. *The space* $\mathbf{M}(\theta)$ *has dimension 1 for all* $\theta \in \mathbb{C} \cup \infty$.

Proof. Suppose $\theta = \infty$. Then $\mathbf{M}(\theta)$ has basis A_D and therefore has dimension 1. Suppose $\theta \in \mathbb{C}$. Then $\mathbf{M}(\theta)$ has dimension 1 by Corollary 6.3. $\square$

Lemma 6.5. *Let* θ *and* θ' *denote distinct elements of* $\mathbb{C} \cup \infty$. *Then* $\mathbf{M}(\theta) \cap \mathbf{M}(\theta') = 0$.

Proof. This is a routine consequence of Corollary 6.3 and the fact that $\mathbf{M}(\infty) = \mathbb{C}A_D$. $\square$

Corollary 6.6. *For* $0 \leq i \leq D$ *we have* $\mathbf{M}(\theta_i) = \mathbb{C}E_i$.

Proof. Observe $(A - \theta_i I)E_i = 0$ so $E_i \in \mathbf{M}(\theta_i)$. The space $\mathbf{M}(\theta_i)$ has dimension 1 by Corollary 6.4 and E_i is nonzero so E_i is a basis for $\mathbf{M}(\theta_i)$. $\square$

Remark 6.7. *[2, p. 63] For* $0 \leq j \leq D$ *we have*

$$E_j = m_j |X|^{-1} \sum_{i=0}^{D} f_i(\theta_j) k_i^{-1} A_i,$$

where m_j *denotes the rank of* E_j.

Definition 6.8. Let $\theta \in \mathbb{C} \cup \infty$. By a *pseudo primitive idempotent* for θ we mean a nonzero element of $\mathbf{M}(\theta)$, where $\mathbf{M}(\theta)$ is from Definition 6.1.

7 The local eigenvalues

Definition 7.1. Define a function $\widetilde{\ }: \mathbb{C} \cup \infty \longrightarrow \mathbb{C} \cup \infty$ by

$$\widetilde{\eta} = \begin{cases} \infty & \text{if } \eta = -1, \\ -1 & \text{if } \eta = \infty, \\ -1 - \frac{b_1}{1+\eta} & \text{if } \eta \neq -1, \eta \neq \infty. \end{cases}$$

Observe $\widetilde{\widetilde{\eta}} = \eta$ for all $\eta \in \mathbb{C} \cup \infty$.

Let v denote a nonzero vector in E_1^*V which is orthogonal to s_1. Assume v is an eigenvector for $E_1^*AE_1^*$ and let η denote the corresponding eigenvalue. We recall a few facts concerning η and $\widetilde{\eta}$. We have $\widetilde{\theta}_1 \leq \eta \leq \widetilde{\theta}_D$ [18, Theorem 1]. If $\eta = \widetilde{\theta}_1$ then $\widetilde{\eta} = \theta_1$. If $\eta = \widetilde{\theta}_D$ then $\widetilde{\eta} = \theta_D$. We have $\theta_D < -1 < \theta_1$ by [18, Lemma 3] so $\widetilde{\theta}_1 < -1 < \widetilde{\theta}_D$. If $\widetilde{\theta}_1 < \eta < -1$ then $\theta_1 < \widetilde{\eta}$. If $-1 < \eta < \widetilde{\theta}_D$ then $\widetilde{\eta} < \theta_D$. We will show that if $\widetilde{\theta}_1 < \eta < \widetilde{\theta}_D$ then $\widetilde{\eta}$ is not an eigenvalue of Γ. Given the above inequalities, to prove this it suffices to prove the following result.

Proposition 7.2. *Let v denote a nonzero vector in E_1^*V. Assume v is an eigenvector for $E_1^*AE_1^*$ and let η denote the corresponding eigenvalue. Then $\widetilde{\eta} \neq k$.*

Proof. Suppose $\widetilde{\eta} = k$. Then $\eta = \widetilde{k}$ so by Definition 7.1,

$$\eta = -1 - \frac{b_1}{k+1}.$$

By this and since $b_1 < k$ we see η is a rational number such that $-2 < \eta < -1$. In particular η is not an integer. Observe η is an eigenvalue of the subgraph of Γ induced on the set of vertices adjacent x; therefore η is an algebraic integer. A rational algebraic integer is an integer so we have a contradiction. We conclude $\widetilde{\eta} \neq k$. $\square$

Corollary 7.3. *Let v denote a nonzero vector in E_1^*V which is orthogonal to s_1. Assume v is an eigenvector for $E_1^*AE_1^*$ and let η denote the corresponding eigenvalue. Suppose $\widetilde{\theta}_1 < \eta < \widetilde{\theta}_D$. Then $\widetilde{\eta}$ is not an eigenvalue of Γ.*

8 The proof of Theorem 1.2

We now give a proof of Theorem 1.2.

Proof. We first show E is contained in $(\mathbf{M}; v)$. To do this we show $Ev \in E_D^*V$. First suppose $\eta \neq -1$. Then $\widetilde{\eta} \in \mathbb{C}$ by Definition 7.1. By Definition 6.1 there exists $\epsilon \in \mathbb{C}$ such that $(A - \widetilde{\eta}I)E = \epsilon A_D$. By this and Lemma 3.1(ii),

$$\begin{aligned} AEv &= \widetilde{\eta}Ev + \epsilon A_D v \\ &\in \mathbb{C}Ev + E_{D-1}^*W + E_D^*W. \end{aligned} \tag{8.1}$$

In order to show $Ev \in E^*_D V$ we show $E^*_i Ev = 0$ for $0 \le i \le D-1$. Observe $E^*_0 Ev = 0$ since $E^*_0 Ev \in E^*_0 W$ and W has endpoint 1. We show $E^*_1 Ev = 0$. By Corollary 6.3 there exists a nonzero $m \in \mathbb{C}$ such that

$$E = m \sum_{h=0}^{D} f_h(\widetilde{\eta}) k_h^{-1} A_h.$$

Let us abbreviate

$$\rho_h = m f_h(\widetilde{\eta}) k_h^{-1} \qquad (0 \le h \le D), \tag{8.2}$$

so that $E = \sum_{h=0}^{D} \rho_h A_h$. By this and (3.2) we find $E^*_1 E E^*_1 = \sum_{h=0}^{2} \rho_h E^*_1 A_h E^*_1$. Applying this to v we find

$$E^*_1 Ev = \sum_{h=0}^{2} \rho_h E^*_1 A_h v. \tag{8.3}$$

Setting $i = 1$ in Lemma 3.1(i), applying each term to v, and using $Jv = 0$ we find

$$0 = \sum_{h=0}^{2} E^*_1 A_h v. \tag{8.4}$$

By (8.3), (8.4), and since $E^*_1 Av = \eta v$ we find $E^*_1 Ev = \gamma v$ where $\gamma = \rho_0 - \rho_2 + \eta(\rho_1 - \rho_2)$. Evaluating γ using (2.6), (8.2), and Definition 7.1 we routinely find $\gamma = 0$. Apparently $E^*_1 Ev = 0$. We now show $E^*_i Ev = 0$ for $2 \le i \le D-1$. Suppose there exists an integer j $(2 \le j \le D-1)$ such that $E^*_j Ev \ne 0$. We choose j minimal so that

$$E^*_i Ev = 0 \qquad (0 \le i \le j-1). \tag{8.5}$$

Combining this with (8.1) we find

$$E^*_i AEv = 0 \qquad (0 \le i \le j-1). \tag{8.6}$$

Since W is thin and since $E^*_j Ev \ne 0$ we find $E^*_j Ev$ is a basis for $E^*_j W$. Apparently $E^*_{j-1} AE^*_j Ev$ spans $E^*_{j-1} AE^*_j W$. The space $E^*_{j-1} AE^*_j W$ is nonzero by (4.2) and since the diameter of W is at least $D-2$. Therefore $E^*_{j-1} AE^*_j Ev \ne 0$. We may now argue

$$\begin{array}{rcll} E^*_{j-1} AEv & = & \sum_{i=0}^{D} E^*_{j-1} AE^*_i Ev & \\ & = & E^*_{j-1} AE^*_j Ev & \quad \text{by (3.3), (8.5)} \\ & \ne & 0 & \end{array}$$

which contradicts (8.6). We conclude $E_i^*Ev = 0$ for $2 \leq i \leq D-1$. We have now shown $E_i^*Ev = 0$ for $0 \leq i \leq D-1$ so $Ev \in E_D^*V$ in the case $\eta \neq -1$. Next suppose $\eta = -1$, so that $\widetilde{\eta} = \infty$. By Definition 6.1 there exists a nonzero $t \in \mathbb{C}$ such that $E = tA_D$. In order to show $Ev \in E_D^*V$ we show $A_Dv \in E_D^*V$. Since A_Dv is contained in $E_{D-1}^*V + E_D^*V$ by Lemma 3.1(ii), it suffices to show $E_{D-1}^*A_Dv = 0$. To do this it is convenient to prove a bit more, that $E_i^*A_{i+1}^*v = 0$ for $1 \leq i \leq D-1$. We prove this by induction on i. First assume $i = 1$. Setting $i = 1$ in Lemma 3.1(i), applying each term to v and using $Jv = 0$, $E_1^*Av = -v$, we obtain $E_1^*A_2^*v = 0$. Next suppose $i \geq 2$ and assume by induction that $E_{i-1}^*A_iv = 0$. We show $E_i^*A_{i+1}v = 0$. To do this we assume $E_i^*A_{i+1}v \neq 0$ and get a contradiction. Note that $E_i^*A_{i+1}v$ spans E_i^*W since W is thin. Then $E_{i-1}^*AE_i^*A_{i+1}v \neq 0$ by (4.2). But $E_{i-1}^*AE_i^*A_{i+1}v = b_iE_{i-1}^*A_iv$ by Lemma 3.4(ii). Of course $b_i \neq 0$ so $E_{i-1}^*A_iv \neq 0$, a contradiction. Therefore $E_i^*A_{i+1}^*v = 0$. We have now shown $E_i^*A_{i+1}^*v = 0$ for $1 \leq i \leq D-1$ and in particular $E_{D-1}^*A_Dv = 0$. It follows $Ev \in E_D^*V$ for the case $\eta = -1$. We have now shown $Ev \in E_D^*V$ for all cases so $E \in (\mathbf{M}; v)$. We now prove E, J form a basis for $(\mathbf{M}; v)$. By Theorem 1.1 $(\mathbf{M}; v)$ has dimension 2. We mentioned earlier $J \in (\mathbf{M}; v)$. We show E, J are linearly independent. Recall E, J are pseudo primitive idempotents for $\widetilde{\eta}, k$ respectively. We have $\widetilde{\eta} \neq k$ by Proposition 7.2 so E, J are linearly independent in view of Lemma 6.5. $\square$

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